On the root problem in some constructions of Artin groups

Dobrynina Irina Vasiljevna

Doctor of Physico-mathematical Sciences, Full Professor

Civil Defence Academy, EMERCOM of Russia

Abstract. An algorithm for solving the root problem in a tree product of Artin groups with a tree structure and extra-large type is considered.

Keywords: Artin group, tree structure, extra-large type, root problem, diagram.

Consider a group

$$\breve{G} = \langle \sigma_1, \dots, \sigma_m; < \sigma_i\sigma_j >^{m_{ij}} = < \sigma_j\sigma_i >^{m_{ji}}, i, j = \overline{1, m}, i \neq j \rangle,$$

$< \sigma_i\sigma_j >^{m_{ij}}$ – a word from m_{ij} sequentially taken generators $\sigma_i, \sigma_j, i \neq j$, m_{ij} – numbers that are elements of the Coxeter matrix [1].

At $m_{ij} > 3$ for arbitrary, unequal indices, we have an extra large type $\breve{G}$ [1].

If the graph in the image $\breve{G}$ is a tree, then we obtain a tree structure [2].

Consider further the following construction:

$$G = \langle \prod_{k=1}^{p} * G_k; \sigma_{i_t}^{\alpha_{i_t}} = \sigma_{j_s}^{\alpha_{j_s}}, i, j \in \{\overline{1, p}\}, i \neq j \rangle,$$

which is the product of G_k having a tree structure or extra-large type, where $\sigma_{i_t}^{\alpha_{i_t}}$ – is the degree of the generator G_i, $\sigma_{j_s}^{\alpha_{j_s}}$ – is the degree of the generator G_j.

In the factors of this construction, the problems of identity (equality) and conjugacy are solvable [1]-[3].

Consider diagrams over a construction G. The conjugacy of elements will be denoted by the symbol ~.

Theorem 1. The *R*-diagram of the conjugacy of non-powers of generators of cyclically *R*, $\bar{R}$ – irreducible elements of G is one-layer.

The *R*-diagram of the equality of non-powers of the generators of *R*, $\bar{R}$ -irreducible elements of G is one-layer.

The proof of the theorem is similar to the proof presented in [3].

The *R*-diagram of the conjugacy of elements that are cyclically *R* and $\bar{R}$-irreducible is called especially special [2] when there is a single region for which the syllable length of the intersection mark with one of the boundaries by 2 units differs from the syllabic length of the intersection mark with the other of the boundaries. We get that on one of the boundaries the syllable length of the element is less than on the other. Then the replacement of an element with a shorter syllable length is called a special annular contraction [2].

Definition 2 [2]. An element is called dead-end if the special annular cancellation is not applicable to it.

It is possible to construct an algorithm that establishes for any element whether it will be a dead end, which follows from [1] - [3].

Lemma 2. If two dead-end elements are conjugate, then their syllable lengths are equal.

Indeed, if one of the dead-end elements has a large length and is conjugated to a given one, then the abbreviations defined above can be performed in it. This can be easily seen in the contingency diagram of the elements.

Lemma 3. The construction G is torsion-free.

The proof follows from the fact that if an element of G is of finite order, then it must be conjugate to an element of finite order of some factor, but it follows from [4], [5] that the factors are torsion-free. Consequently, it is absent in the construction under consideration.

Lemma 4 [6]. Given a cyclically R, $\bar{R}$-irreducible nontrivial element u, then there exists an element v, for which every power of R, $\bar{R}$-irreducible, and $v \sim u$, or $v \sim u^2$.

Definition 3 [4]. Problem 1 is understood as the problem of constructing an algorithm that determines whether one element is a nonunit power of some other element.

Theorem 2. In the construction of G, Problem 1 is algorithmically solvable.

Lemma 5 [4]. Problem 1 is solvable in tree-structure factors of the construction G.

Lemma 6 [5]. Problem 1 is solvable in factors of construction G of extra-large type.

Let us proceed to the proof of the theorem.

First of all, note that Problem 1 is solvable in the factors of G on the basis of Lemmas 5 and 6. Therefore, we will consider the general case when there are at least two factors.

Suppose that in the construction G under consideration for some elements u^n, w the equality

$$u^n = w. \tag{1}$$

Let us square both sides of equality (1). We get

$$(u^2)^n = w^2. \tag{2}$$

Carrying out the necessary cancellations *in* u we replace it with an element u_0 in accordance with Lemma 4 so that any degree u_0 R, $\bar{R}$-irreducible. Then equality (2) takes the form:

$$u_0^n = z^{-1} w^2 z. \tag{3}$$

The element $z^{-1}w^2z$ is equal to some $R, \bar{R}$-irreducible element w_0. Equality (3) takes the form:

$$u_0^n = w_0. \quad (4)$$

Consider the diagram of equality of elements u_0^n, w_0, corresponding to (4).

Thus, M - is a simply connected diagram as in [3] with a boundary cycle $\partial M = \sigma \cup \delta$, where $\varphi(\sigma) = u_0^n$ and $\varphi(\delta) = w_0$.

By Theorem 1, this diagram will be one-layer. It follows from this that the number of regions having an exit to σ and δ is the same. The edges with an exit to δ are at most than the syllable length of the word v_0 ($||v_0||$). The number of vertices cannot exceed $|v_0| + 1$. Therefore, the number of regions is limited by the exponent $k = ||w_0|| + \left||w_0|\right| + 1 = 2||w_0|| + 1$.

We have $n < ||u_0^n|| < k||r|| \le (2||u_0|| + 1)||r||$, where r – is the longest defining relation, its length is not greater than the number $||w_0|| + ||u_0|| + 2$. Hence $n < (2||u_0|| + 1)(||w_0|| + ||u_0|| + 2)$.

In the case when the diagram consists of disks and simple paths connecting them, the proof of the theorem is obvious.

Thus, the exponent can be limited, and this provides a solution to Problem 1.

Definition 4 [5]. Problem 2 is solvable in a group if, for any of its elements *u*, it is effectively established that there exists a non-trivial natural number *n* and an element *x* satisfying the equality $x^n = u$.

Theorem 3. Problem 2 is solvable in the construction of *G*.

The proof of this theorem follows from the validity of Theorem 1 in the construction *G*.

Corollary 1. In the construction of *G*, any sequence of subgroups generated by one element is stabilized.

The construction *G* considered in this paper is a generalization of the constructions presented in [3], [6]. Indeed, the author considers the union of groups by subgroups generated by the degrees of the generators of the corresponding groups, in the case when the exponents of these degrees are equal to zero, the groups from [3], [6] are obtained. And, therefore, problems 1, 2, 3 considered in this paper are solvable in them.

Definition 5 [6]. Problem 3 is understood as the problem of the existence of an algorithm that, for any two elements of a group that do not belong to the same cyclic subgroup, finds an integer *n* such that the *n*-th power of one element is conjugate in the group to another element.

Theorem 4. Problem 3 is solvable in the construction of *G*.

The proof repeats the proof of Theorem 3 from [6], since, based on Theorem 1, the diagrams have a structure similar to the diagrams considered in [3] of groups.

In this paper, the study of Artin groups by geometric methods is continued. Note that the main algorithmic problems, which include the identity (equality) and conjugacy problems, follow from [7], but the algorithms are cumbersome and more complicated than those proposed in [3]. In the further study of Artin groups, one can consider similar constructions, where other well-known classes of Artin groups, for example, of large or finite type, are taken as factors.

References

1. Appel K., Schupp P. Artins groups and infnite Coxter groups // Ivent. Math. 1983. V. 72. P. 201-220.
2. Bezverkhny V.N., Karpova O. Yu. Problems of equality and conjugacy of words in Artin groups with tree structure. Izvestiya TulGU. Series Mathematics. Mechanics. Computer science. 2006. V. 12, № 1. P. 67–82.
3. Dobrynina I.V., Ugarov A.S. On generalized tree structures of Artin groups // Vladikavkaz mathematical journal.2021. V. 23, №3. P.52-63.
4. Bezverkhny V.N. and Karpova O. Yu., "The problem of entering a cyclic subgroup in Artin groups with a tree structure," Chebyshevskii Sbornik.2008. V.9, №1. P. 30-49.
5. Bezverkhny V.N., Kuznetsova A.N. The problem of the root in large-type Artin groups // L. Euler and Russian education, science and culture: Materials of the international scientific-practical conference. Tula, 2007. P. 36-43.
6. Dobrynina I. V. On the problem of weak power conjugacy in a special class of Artin groups // Proceedings of the International Conference «Process Management and Scientific Developments» (Birmingham, United Kingdom, September 1, 2021). P. 220-224.
7. Derek F. Holt, Sarah Rees. Biatomatic structures in systolic Artin groups. 2018. arXiv:1807.02532v1 [math. GR].