On the application of partial operations for the description of information systems

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Abstract

The representation of information systems is based on various algebraic methods. The algebraic theory of graphs, including partial graphs, is based on the use of algebraic means, and therefore there is an interest in the detailed study of the theory of partial algebraic actions.

Keywords

Partial graphs, graphs, semigroups, bicyclic semigroups, algebraic actions, congruence, homomorphism.

Abstract description of information systems [7] is a common method for mathematical, logical description of the functioning of information systems. This description is represented by a set of levels of abstract description, and the generally accepted levels are: symbolic (linguistic), set-theoretic, abstract algebraic, topological, logical-mathematical, information-theoretical, dynamic, heuristic (intuitive). Abstract description of the information system [7] at the appropriate levels allows: to evaluate the indicators that characterize various properties of the IP; to choose the optimal system structures; to select and maintain optimal values of the IP parameters; to solve other problems of quality assurance and optimization.

As you know, one of the ways to study a particular algebraic system is to decompose it into subsystems from a sufficiently studied class. In the theory of semigroups, expansions into the union of pairwise disjoint sub-semigroups or, sometimes, pairwise intersecting at a common zero are widely used. The works of A. Clifford, V. Mann, M. Petrich, R. Croiseau, D. Howie, L. N. Shevrin, A.V. Kelarev and many others are known in this direction.

The main object of the study is the class of categorical semigroups that allow decompositions into the union of Brandt semigroups with a common zero. Note that every statement about semigroups with zero implies as an obvious consequence some statement about semigroups without zero, if we assume that in the semigroup under consideration zero is external.

The study of semigroups that are the 0-union of Brandt semigroups seems relevant, since in the class of semigroups with zero, the Brandt semigroup is the most natural analogue of the concept of a group. For example, Wechler and Fichtner use Brandt and Ehresmann groupoids to describe the symmetry of crystals, and the zero extension of the fundamental groupoid of any undirected graph is also a direct union of Brandt semigroups. Another example. Let $M=\{M_i \mid i\in I\}$ be the set of pairwise non-intersecting non-empty sets. Then the set of all bijections whose domain of definition and domain of value belong to M (these domains may coincide), with respect to the usual superposition of maps, is a partial groupoid whose zero extension is a semigroup that is the 0-union of Brandt semigroups. For example, as M, we can take – the set of open faces (without edges) of a polyhedron, in particular, - some crystal.

The formulations of the results obtained become much shorter, and their proofs are significantly simplified if, instead of the semigroup with zero under study, we consider the partial groupoid that is obtained from this semigroup by removing zero.

The main method of research in this work consists in using an operation on classes of partial groupoids, which is close to multiplying classes of complete groupoids, first considered by A. I. Maltsev. In these terms, we can also consider the concept of graded algebra.

To solve this problem on partial $\mathfrak{I}$ groupoids with certain associativity-type conditions, we study congruences whose adjacent classes are Brandt groupoids. On the partial groupoids we study, the only congruence satisfying this requirement is the Green equivalence . By revealing the various properties of this equivalence and the subsequent transition to the zero extension of the considered partial groupoids, the goal set in this paper is achieved: the structure of categorical semigroups at zero, which are the 0-union of Brandt semigroups in terms of partial semilattices , is described.

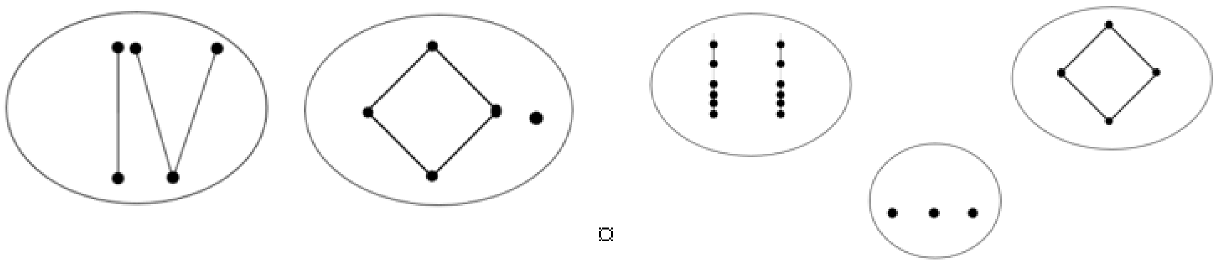

Figure 1 – Examples of partial semilattices

The concept of a catenary semigroup was introduced for geometric purposes by V. V. Wagner [1]. A semigroup is a strongly associative partial groupoid $(S;\cdot)$ i.e. a partial groupoid is a semigroup if and only if one of the products $(x\cdot y)\cdot z$, $x\cdot(y\cdot z)$ is defined in S , it follows that the other is also defined, and the equality holds

$$(x\cdot y)\cdot z = x\cdot(y\cdot z) \tag{1}$$

If the product $u\cdot v$ is not defined in S, then we write $u\cdot v=\varnothing$.

Obviously, a partial groupoid $(S;\cdot)$ is a semigroup if and only if its zero extension [2] $S^0 = S\cup\{0\}$ is a semigroup. Therefore, every property of semigroups entails an obvious consequence for semigroups (with zero).

An idempotent commutative semigroup is called a partial semilattice.

An idempotent partial groupoid is called an antichain if the condition $xy\neq\varnothing$ always implies $x=y$

A partial groupoid $(S;\cdot)$ is called catenary if the condition $x\cdot y\neq\varnothing\neq y\cdot z$ always implies $(x\cdot y)\cdot z\neq\varnothing\neq x\cdot(y\cdot z)$.

The condition of catenarity of a partial groupoid of a groupoid S is equivalent to the condition of categoricity at zero of its zero extension S^0.

The accepted semigroup terminology is preserved [6] for arbitrary semigroups as well. For example, the terms regular, inverse, simple, completely simple semigrupoids, etc. are clear. A completely simple inverse semigrupoid is called a Brandt groupoid.

For arbitrary classes Σ,Γ of semigrupoids, we denote $\Sigma*\Gamma$ by the class of all semigrupoids S on which there exists such a congruence τ that $S/\tau\in\Gamma$, and every closed in S τ-class belongs to Σ. Every semigrupoid of the class $\Sigma*\Gamma$ is called Γ-semigrupoid of Σ-semigrupoids. The paper considers the structure of catenary semigrupoids, which are idempotent commutative semigrupoids of Brandt groupoids. A special case of the result obtained in this work is one of the main theorems in [4].

Notation: $\boldsymbol{A}$ – class of antichains; $\boldsymbol{B}$ - class of Brandt groupoids; $\boldsymbol{I}$ -class of semilattices; $\boldsymbol{Q}$ -class of catenary partial semilattices; $\boldsymbol{M}$ -class of inverse Clifford semigrupoids in which the ideal Green $\mathfrak{I}$ equivalence is a congruence; $\boldsymbol{K}$ is the class of those semigrupoids $S\in\boldsymbol{M}$ for which the binary relation

$$\lhd=\{(\alpha,\beta)\in S/\mathfrak{I}\times S/\mathfrak{I} \mid \alpha\circ\beta=\alpha\}$$

satisfies the condition

$$(\gamma \lhd \alpha \,\&\, \gamma \lhd \beta) \rightarrow (\gamma \lhd \alpha \circ \beta) \qquad (2)$$

Theorem 1.

$$\boldsymbol{Q} = \boldsymbol{I} * \boldsymbol{A}$$

It is proved in [3] that semigrupoids of class $\boldsymbol{K}$ and only they are catenary partial semilattices of Brandt groupoids, i.e. $\boldsymbol{K}=\boldsymbol{B}*\boldsymbol{Q}$, whence, by Theorem 1, $\boldsymbol{K}=\boldsymbol{B}*(\boldsymbol{I}*\boldsymbol{A})$. Naturally, the question arises about the structure of catenary semigrupoids of class $\boldsymbol{K}$. The solution of this question is the purpose of this work.

Theorem 2.

A semigroup S of class $\boldsymbol{K}$ is catenary if and only if for any $\alpha, \beta \in S/\mathfrak{I}$, such that $\beta \lhd \alpha$ and any idempotent $e \in \alpha$ there is a unique idempotent $f \in \beta$ such that $f \leq e$.

The solution of this problem in a purely semigroup language presents significant difficulties. This is caused by the following circumstance. The decomposition of a semigroup *S* into sub-semigroups with a common zero does not determine not only congruences on *S*, but even equivalences. An attempt to isolate zero, considering it a separate class, is untenable: the decompositions of S under consideration are such that the binary relations corresponding to them on the partial groupoid $S\backslash\{0\}$, being congruences, are not strong congruences, and therefore their zero extensions (using the pair (0,0)) are not congruences on the semigroup S. That is why the language of partial actions is preferable to the language of complete actions.

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