Supercritical convection in a porous medium

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Abstract. Convection in a heat-insulated layer with heating from below is investigated. The approximation is considered large in comparison with the thickness of the layer length. The parameters of instability are determined.

Keywords: Rayleigh number, layer, simulation

Introduction

The foundations of the theory of convection are presented in [1]. General cases of supercritical convection are considered in [2]. The stability of the convective equilibrium of a liquid in a layer heated from below in the case when the thermal conductivity of the masses that bound the layer is finite was studied in [3]. Harl, Jakeman, and Pike considered the symmetric case of arrays of equal thermal conductivity; the boundary value problem was solved for neutral, non-increasing and non-decreasing perturbations in time, and was approximately solved by the Galerkin method. A general formula was obtained for the critical Rayleigh number depending on the wavenumber and the ratio of the thermal conductivity of the liquid to the thermal conductivities of the surrounding massifs. It was shown that in the case of a layer with heat-insulating boundaries (for example, liquid metal between glass plates), the critical minimum wavenumber vanishes, the corresponding critical Rayleigh number is 720. At $R > R_0$ perturbations with wavenumbers α that lie in the interval $0 < \alpha < \alpha_m(R)$

In [4], the onset of convection in a horizontal layer of a porous medium, considered for the first time by Horton and Rogers, was investigated; the experiments they set up yielded results sharply deviating from theory. Kato and Matsuoka showed that in previous works the coefficient of effective thermal diffusivity of a saturated porous medium was incorrectly determined; their experiments gave good agreement with the revised theory. In these works, the case of a highly heat-conducting array surrounding the layer was considered. In this work, the task was to study the dependence of the occurrence of convection on the thermal conductivity of the massif. For a layer with ideally heat-conducting boundaries, the minimum of the neutral curve lies at $a = a_c \neq 0$, so that for small excess of R_0 the instability interval has the form $a_1 < a < a_2$, where a_1, a_2 are close to a_c. In this case, perturbations with multiple wavenumbers $n\,a\,(n = 0,2,3\ldots)$ decay according to the linear theory. The most dangerous are long-wave disturbances. The solution can be constructed as a series in terms of the amplitude of the fundamental harmonic. By analogy with a homogeneous liquid, they were limited to the long-wave approximation; for small $R-R_0$ the value of a_m, is small, the velocity, temperature, and Rayleigh number in the system of amplitude equations for neutral perturbations were expanded in powers of the wave number. The expansion of the Rayleigh number was obtained up to the second order:

$$R_p = 12 + \frac{8}{7} a_m^2 + 0\left(a_m^4\right)$$

and also the general formula for the Rayleigh number.

In [5], the convection of a liquid in a layer of a homogeneous medium was investigated taking into account the nonlinear terms in the equations. A two-dimensional case was considered, the stream and temperature functions depended on the vertical and one of the horizontal coordinates; small excesses of R_0. The method of a small parameter was used, the stream functions, temperatures and the Rayleigh number were expanded in a series in powers of the wave number. The expansion of the Rayleigh number up to the second order is obtained:

$$R = 720 + \frac{2\,040}{77} a_m^2 + 0\left(a_m^4\right)$$

An equation for the first term in the temperature expansion is obtained

$$\frac{\partial N}{\partial \tau} + \frac{\partial^4 N}{\partial \xi^4} + \frac{\partial^2 N}{\partial \xi^2} + \frac{\partial^2}{\partial \xi^2}\left(N^3\right)$$

where $N \square \frac{\partial \mathrm{T}_0}{\partial \xi}$

and its solution was found for the stationary case; it was also shown using the small parameter method that this motion is unstable with respect to long-wave disturbances.

In this work, convection in a horizontal layer of a porous medium with heat-insulating surrounding massifs is investigated. The nonlinear gradient term is taken into account in the heat conduction equation, three-dimensional motions are considered. Small excesses of the critical Rayleigh number are considered. In this work, the small parameter method is used; at small supercriticalities $R-R_0$ the wave number, functions of temperature, pressure, and the critical Rayleigh number are expanded in a series in powers of the wave number (it is convenient to exclude the velocity from the equations):

$$R_p = \sum_{n=0}^{\infty} a_m^{2n} R_{2n}; \quad \mathrm{T} = \sum_{n=0}^{\infty} a_m^4 \mathrm{T}_n; \quad \mathrm{P} = \sum_{n=0}^{\infty} a_m^4 \mathrm{P}_n \tag{1}$$

Porous environment

We add nonlinear terms to the convection equations in a porous medium, then the system has the

form:
$$\begin{cases} \nabla p - \vec{U} + R_p \mathrm{T}\vec{\gamma} = 0 \\ -\frac{\partial \mathrm{T}}{\partial t} + \Delta \mathrm{T} + \vec{U}\vec{\gamma} - \vec{U}\nabla \mathrm{T} = 0 \\ dw\vec{U} = 0 \end{cases} \tag{2}$$

where $\vec{\gamma}$ - unit vector against $\vec{g}$, and

$$R_p = \frac{g\beta L^2 \kappa \mathrm{A}}{\nu\chi}$$

T – temperature deviation from equilibrium distribution $\mathrm{T} = -y + const$; k – permeability coefficient, χ – thermal diffusivity of liquid, ν – kinematic viscosity; unit of length L – layer width, speed is measured in χ/L, pressure - in $\rho_0 \nu\, \chi/\kappa$.

Let us choose a suitable coordinate system, then the boundary conditions for the heat-insulated layer will be written in the form

$$\mathrm{Z} = 0,\ \mathrm{Z} = 1 \quad \mathrm{U}_z = \frac{\partial \mathrm{T}}{\partial z} = 0$$

Eliminating the speed from the equations, we get the system:

$$\begin{cases} -\dfrac{\partial \mathrm{T}}{\partial t} + \Delta \mathrm{T} + \nabla p \nabla \mathrm{T} - R\mathrm{T}\dfrac{\partial \mathrm{T}}{\partial z} + \dfrac{\partial \mathrm{P}}{\partial z} + R\mathrm{T} = 0 \\ -\Delta p + R\dfrac{\partial \mathrm{T}}{\partial z} = 0 \end{cases} \tag{3}$$

for this system, the boundary conditions will be:

$$\mathrm{Z} = 0;\ \mathrm{Z} = 1 \quad \frac{\partial \mathrm{T}}{\partial z} = 0; \quad \nabla \mathrm{P} = -R\mathrm{T}\vec{\gamma} \quad \text{or} \quad \frac{\partial \mathrm{P}}{\partial z} = -R\mathrm{T}$$

An inhomogeneous system of differential equations has a solution if the solvability condition is satisfied [6]; For a given system, the solvability condition is obtained by integrating the expression for $\dfrac{\partial^2 \mathrm{T}_\kappa}{\partial z^2}$, obtained in the κ-th order, over the layer thickness.

To consider processes with different time and space scales, similar to work [4], the method of many scales is used:

functions of pressure and temperature can be represented as depending on a set of variables $t_n = a_m^n t$, where t_1, t_2 etc. – slow times; then the time derivative is:

$$\frac{\partial}{\partial t} = \sum_{n=0}^{\infty} a_m^n t_n$$

(in what follows, we assume that all fast processes have already passed and the functions T, P do not depend, at least, on t_0).

For spatial variables (horizontal), we restrict ourselves to the first order in the series, assuming that in the zero order the functions T, P have no dependence on χ_0, y_0:

$$\xi = a_m \chi, \quad \eta = a_m y$$

Then the derivatives with respect to these coordinates are:

$$\frac{\partial}{\partial \chi} = a_m \frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial y} = a_m \frac{\partial}{\partial \eta}$$

Let us substitute these derivatives and expansion (1) into the system of equations and into the system of equations **Ошибка! Источник ссылки не найден.**. Using the boundary conditions, in order zero, we obtain

$$\mathrm{P}_0 = R_0 \mathrm{T}_0 \cdot z - \frac{R_0 \mathrm{T}_0}{2}; \quad \frac{\partial \mathrm{T}_0}{\partial z} = 0$$

In the first order, we see that the functions T, P do not depend on t_1;

$$\mathrm{P}_1 = R_0\mathrm{T}_1 \cdot z - \frac{R_0\mathrm{T}_1}{2}; \quad \frac{\partial \mathrm{T}_1}{\partial z} = 0$$

In the second order, using the solvability condition, we obtain: $R_0 = 12$ in the same way as in [4]; functions do not depend on t_2;

$$\mathrm{P}_2 = \frac{R_0}{2}\Delta_2\mathrm{T}_0\left(\frac{z^2}{2} - \frac{2}{3}z^3 + \frac{z^4}{2} - \frac{z^5}{5}\right) + \frac{R_0}{2}\left[\nabla_2\mathrm{T}_0\right]^2\left(2z^3 - z^4\right) + R_2\mathrm{T}_0 z + C_2$$

where Δ_2, ∇_2 - two-dimensional Laplacian and Gradient, C_2 - constant of integration independent of Z,

$$\mathrm{T}_2 = \Delta_2\mathrm{T}_0\left(-\frac{z^2}{2} + z^3 - \frac{z^4}{2}\right) + \left(\nabla_2\mathrm{T}_0\right]^2\left(3z^2 - 2z^3\right)$$

In the third order, the calculations are the same as in the second order. We get:

$$\mathrm{P}_3 = \frac{R_0}{2}\Delta_2\mathrm{T}_1\left(\frac{z^2}{2} - \frac{2}{3}z^3 + \frac{z^4}{2} - \frac{z^5}{5}\right) + \frac{R_0}{2}\left[\nabla_2\mathrm{T}_1\right]^2\left(2z^3 - z^4\right) + R_2\mathrm{T}_1 z + C_3$$

$$\mathrm{T}_3 = \Delta_2\mathrm{T}_1\left(-\frac{z^2}{2} + z^3 - \frac{z^4}{2}\right) + \left[\nabla_2\mathrm{T}_1\right]^2\left(3z^2 - 2z^3\right)$$

the functions T, P do not depend on t_3.

In the fourth order, using the solvability condition, we obtain a closed equation for T_0:

$$\frac{\partial \mathrm{T}_0}{\partial t} + \frac{2}{21}\Delta_2^2\mathrm{T}_0 + \frac{R^2}{12}\Delta_2\mathrm{T}_0 - \frac{6}{5}\left(\nabla_2\mathrm{T}_0\nabla_2\right)\left[\nabla_2\mathrm{T}_0\right]^2 - \frac{6}{5}\Delta_2\mathrm{T}_0\left[\nabla_2\mathrm{T}\right]^2 = 0 \quad (2)$$

Since the solution to the linear problem should have the form

$$\mathrm{T}_0 = \exp(i\xi)\exp(i\eta)$$

then from this we obtain the value of $R_2 = \frac{8}{7}$ similarly [3].

To get rid of the coefficients in equation (2), we can introduce the variables

$$\tau = \frac{2}{\sqrt{35}}t, \quad \bar{\mathrm{T}} = \sqrt{\frac{63}{5}} \cdot \mathrm{T}_0$$

then equation (2) will be written in the form:

$$\frac{\partial \bar{\mathrm{T}}}{\partial \tau} + \Delta_2^2\bar{\mathrm{T}} + \Delta_2\bar{\mathrm{T}} - \left(\nabla_2\bar{\mathrm{T}}\nabla_2\right)\left[\nabla_2\bar{\mathrm{T}}\right]^2 - \Delta_2\bar{\mathrm{T}}\left[\nabla_2\bar{\mathrm{T}}\right]^2 = 0 \quad (3)$$

which can be written in the form:

$$\dot{\mathrm{T}} = -div\left(\vec{g} + \Delta_2\vec{g} - \vec{g}g^2\right)$$

where $\vec{g} = \nabla_2\bar{\mathrm{T}}$.

In a plane problem, when only one horizontal coordinate is involved, the equation transforms into the following:

$$\frac{\partial \mathrm{N}}{\partial \tau}+\frac{\partial^4 \mathrm{N}}{\partial \xi^4}+\frac{\partial^2 \mathrm{N}}{\partial \xi^2}-\frac{\partial^2}{\partial \xi^2}\left(\mathrm{N}^3\right)=0 \qquad (6)$$

where $\mathrm{N}=\frac{\partial \bar{\mathrm{T}}}{\partial \xi}$, obtained in [5], the same for porous and homogeneous media.

It was shown in [3] that, in the stationary case, equation **Ошибка! Источник ссылки не найден.** has a solution

$$\mathrm{N}_{CT.}=\sqrt{\frac{2k^2}{1+k^2}}\sin\frac{\xi-\xi_0}{\sqrt{1+k^2}}$$

where k – modulus of the Jacobi elliptic function related to the wave number $\bar{a}=a/a_m$ by the relation

$\bar{a}=\frac{\pi}{2\mathrm{K}(k)\sqrt{1+k^2}}$, where $\mathrm{K}(k)$ - complete elliptic integral of the first kind, ξ_0 - the constant of integration, hereinafter assumed to be zero. In [3], the stability of stationary motion **Ошибка! Источник ссылки не найден.** with respect to long-wave two-dimensional, longitudinal perturbations was also investigated:

$$\mathrm{N}=\mathrm{N}_{CT.}+\Phi(\xi)\cdot e^{i\beta\xi}\cdot e^{-\lambda^{\mathrm{H}}\tau}$$

The small parameter method was used λ^{H} and $\Phi(\xi)$ were represented as an expansion in powers of the small wavenumber β. It was shown that the motion is unstable with respect to these perturbations, and in the second order

$$\lambda_2^{\mathrm{H}}=-\frac{\left(1-k^2\right)^2}{\left(1+k^2\right)\left(2\mathrm{E}(k)/\mathrm{K}(k)-1+k^2\right)}<0$$

for $0<k<1$; where $\mathrm{K}(k),\mathrm{E}(k)$ - complete elliptic integrals of the first and second kind.

We can consider the stability of this plane motion with respect to long-wave three-dimensional perturbations, in the case when the wave vector is directed perpendicular to the motion, i.e. the perturbation is periodic in the coordinate η:

$$\bar{\mathrm{T}}=\mathrm{T}_{стац.}+\tilde{\mathrm{T}}\cdot e^{-\lambda\tau} \qquad (4)$$

$$\tilde{\mathrm{T}}=f(\xi)e^{i\beta\eta}$$

$$\mathrm{T}_{стац,}=\int \mathrm{N}_{CT.}d\xi$$

Substituting (4) into equation (3), we get:

$$-\lambda\tilde{\mathrm{T}}+\Delta_2^2\tilde{\mathrm{T}}+\left(1-\mathrm{N}^2\right)\Delta_2\tilde{\mathrm{T}}=3\partial_\xi(\mathrm{N})^2\cdot\partial_\xi\tilde{\mathrm{T}}+2\mathrm{N}^2\partial_\xi^2\tilde{\mathrm{T}}$$

Substituting the expression for $\tilde{T}$ into the resulting equation, we get:

$$-\lambda f+\left(\partial_\xi^2-\beta^2\right)^2 f-\left[\partial_\xi^2-\beta^2\left(1-\mathrm{N}^2\right)\right]f=3\partial_\xi\left[\mathrm{N}^2\partial_\xi f\right] \qquad (5)$$

Since N^2 - is a function with a period that is half that of N, then the period $f(\xi)$, according to Bloch's theorem, is also half that:

$$f\left(\xi+\pi/\bar{a}\right)=f\left(\xi\right)$$

Then the solvability condition for the expansion in terms of a small parameter of equation (5) in the k-th order will be obtained if in this order the equation is integrated in the range from 0 to $\pi/\bar{a}$ and the derivatives of $f\left(\xi\right)$ are equated to zero; $\lambda,\ f\left(\xi\right)$ are expanded in powers of the small wave number β, as in [3]:

$$f\left(\xi\right)=\sum_{n=0}^{\infty} f_n\beta^n;\quad \lambda=\sum_{n=0}^{\infty}\lambda_n\beta^n$$

Substituting the expansion into equation (5), in the zeroth order we have:

$$-\lambda_0 f_0+\frac{\partial^4 f_0}{\partial\xi^4}+\frac{\partial^2 f}{\partial\xi^2}=3\partial_\xi\left[\mathrm{N}^2\partial_\xi f_0\right]$$

or by designating $\Phi_0=\partial_\xi f_0$, we get:

$$-\lambda_0\Phi_0+\partial_\xi^4\Phi_0+\partial_\xi^2\left[\left(1-3\mathrm{N}^2\right)\Phi_0\right]=0$$

This equation coincides with the equation obtained in [3]; the most interesting case is $\lambda_0=0$, then, as shown in [4], the equation has a solution:

$$\Phi_0=1-k^2+2k^2cn^2\frac{\xi}{\sqrt{1+k^2}}$$

In the first order, we get a similar equation:

$$-\lambda_1 f_0+\partial_\xi^4 f_1+\partial_\xi^2 f_1=3\partial_\xi\left[\mathrm{N}^2\partial_\xi f_1\right]$$

In this order for $\lambda_1=0$ there is also a solution

$$f_1=f_0+const$$

In the second order, we get the equation:

$$-\lambda_2 f_0+\partial_\xi^4 f_2-2\partial_\xi^2 f_0+\partial_\xi^2 f_2-\left(1-\mathrm{N}^2\right)f_0=3\partial_\xi\left[\mathrm{N}^2\partial_\xi f_2\right]$$

From the solvability condition, we find an expression for λ_2:

$$\lambda_2=-\frac{\int_0^{\pi/\bar{a}}\left(1-\mathrm{N}^2\right)f_0 d\xi}{\int_0^{\pi/\bar{a}} f_0 d\xi}$$

Explicitly substituting in it $f_0=\int\Phi_0 d\xi$, and putting the constant of integration equal to zero, we get:

$$\lambda_2=-1+\frac{2k^2}{1+k^2}\cdot\frac{\int_0^{\pi/\bar{a}} d\zeta\sin^2\zeta\left[2\mathrm{E}\left(am\zeta,k\right)-\left(1-k^2\right)\sqrt{1+k^2}\cdot\zeta\right]}{\int_0^{\pi/\bar{a}} d\zeta\left[2\mathrm{E}\left(am\zeta,k\right)-\left(1-k^2\right)\sqrt{1+k^2}\cdot\zeta\right]}$$

where $\zeta=\frac{\xi}{\sqrt{1+k^2}}$. Compare λ_2 to λ_2^{H}. Numerical estimates up to two decimal places for $k=1/2$ show that $\lambda_2^{\mathrm{H}}=-0,36;\ \lambda_2=-0,47$. I.e. $\lambda_2<\lambda_2^{\mathrm{H}}$, hence, the instability of motion

Ошибка! Источник ссылки не найден. is stronger relative to long-wave transverse three-dimensional perturbations than relative to plane longitudinal.

Convection in a layer with a thickness much less than the length

Equations with nonlinear terms are considered; two-dimensional motions with a small wavenumber are investigated; at small excess of the critical Rayleigh number. The calculations are performed in the same way as in [5], the convection equations are solved using the small parameter method; in this case, a small parameter is the ratio of the layer thickness to its length. In addition, since the medium is porous, the order of the equations decreases by 2, but the boundary conditions also become smaller - on the walls of the massif, the horizontal component of the averaged velocity is not zero. However, the solution is similar to the solution for a homogeneous liquid; the stream function Ψ is also introduced; in the case of an unbounded layer, in the fourth order, a closed equation for $\mathrm{N} \equiv \frac{\partial \mathrm{T}_0}{\partial \xi}$, is obtained, which is the same for porous and homogeneous media:

$$\frac{\partial \mathrm{N}}{\partial \tau}+\frac{\partial^4 \mathrm{N}}{\partial \xi^4}+\frac{\partial^2 \mathrm{N}}{\partial \xi^2}-\frac{\partial^2}{\partial \xi^2}\left(\mathrm{N}^3\right)=0$$

In the case of a bounded layer, the calculations are the same, but

$$R=R_0+R_2 / L^2; R_2=(R-R_0) / L^2$$

where L – layer length, measured in layer thicknesses. The equation turns out:

$$\frac{\partial \mathrm{N}}{\partial \tau}+\frac{\partial^4 \mathrm{N}}{\partial \xi^4}+\mathrm{K}\frac{\partial^2 \mathrm{N}}{\partial \xi^2}-\frac{\partial^2}{\partial \xi^2}\left(\mathrm{N}^3\right)=0$$

where $\mathrm{K}=\frac{21}{2}L^2\left(\frac{R}{12}-1\right)$,now this quantity plays the role of the Rayleigh number.

The boundary conditions for this equation can be obtained from condition $d\Psi=0$; streamlines are obtained from equation $\Psi=const$, it is convenient to put $const=0$, this will not affect the result.

As in [4], the stream functions are proportional to the derivatives of T_0, T_1:

$$\Psi_1=-\frac{R_0}{2}\frac{\partial \mathrm{T}_0}{\partial \xi}\left(y^2-y\right); \quad \Psi_2=-\frac{R_0}{2}\frac{\partial \mathrm{T}_1}{\partial \xi}\left(y^2-y\right)$$

$$\Psi_3=-\frac{\partial^3 \mathrm{T}_0}{\partial \xi^3}\left(\frac{3}{5}y^5-\frac{y^6}{5}+y^3-y^4\right)+\frac{\partial}{\partial \xi}\left(\frac{\partial \mathrm{T}_0}{\partial \xi}\right)^2\left(3y^4-\frac{6}{5}y^5\right)-\frac{R_2}{2}\frac{\partial \mathrm{T}_0}{\partial \xi}\left(y^2-y\right)+6\frac{\partial \overline{\mathrm{T}}_2}{\partial \xi}\left(y^2-y\right)$$

Since $\mathrm{N} \equiv \frac{\partial \mathrm{T}_0}{\partial \xi}$, then the boundary conditions will be:

$$\xi=0, \quad \xi=1 \quad \mathrm{N}=0; \quad \mathrm{N}^{11} \equiv \frac{\partial^3 \mathrm{T}_0}{\partial \xi^3}=0$$

The solution of the equation is investigated depending on the number κ; the equation is solved numerically by the finite difference method. For convenience, we split the equation into a system:

$$\begin{cases} \dfrac{\partial \mathrm{N}}{\partial \tau} + \dfrac{\partial^2 \varphi}{\partial \xi^2} + \kappa\varphi - 6\left(\partial_\xi \mathrm{N}\right)^2 \mathrm{N} + 3\mathrm{N}^2\Phi = 0 \\ \dfrac{\partial^2 \mathrm{N}}{\partial \xi^2} - \varphi = 0 \end{cases}$$

at $\xi = 0, \xi = 1 \quad \mathrm{N} = \varphi = 0$.

The finite difference scheme for φ has the form:

$$\varphi_i = \frac{\mathrm{N}_{i+1} - 2\mathrm{N}_i + \mathrm{N}_{i-1}}{\mathrm{HX}^2}$$

where HX – coordinate step; the step HX = 0.1 is chosen, i.e. ten knots. An explicit scheme is used:

$$\mathrm{N}_i^{n+1} = \mathrm{N}_i^n - \left[\left(\varphi_{i+1}^n - 2\varphi_i^n + \varphi_{i-1}^n\right) - \left(-\mathrm{K} + 3\mathrm{N}_i^{n^2}\varphi_i^n + \frac{\left(6\mathrm{N}_i^n\left(\mathrm{N}_{i+1}^n - \mathrm{N}_{i-1}^n\right)^2\right)}{\mathrm{HX}^2}\right)\right]\mathrm{HT}$$

In order for the explicit scheme to be stable, it is necessary to take a step in time

$$\mathrm{HT} = \mathrm{HX}^5$$

It is seen from the linear problem that the solution $\mathrm{N} \sim \sin \pi\xi$, passes through, whence the critical number $\kappa = \pi^2$ is obtained.

At $\kappa < \pi^2$ and at κ – slightly exceeding π^2 the initial distribution has the form of an inverted parabola with a small slope.

At $\kappa < \pi^2$ the movement, as you would expect, quickly fades away; at $\kappa > \pi^2$ the motion grows and enters a stationary regime, and the greater the number of κ, the greater the amplitude of the steady motion. The dependence of the sum of $\sum_i \mathrm{N}_i$ over the nodes at the 10^6 step on the number κ shows that if the initial distribution is greater in magnitude than the stationary one, then at $\kappa > \pi^2$ the amplitude of motion decreases to stationary. This work can be used to study convection in stellar-type systems.

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