Studying of Vector Optimization Problem under Condition of Tolerance

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Annotation.

In the given work the problem of decision-making under condition of tolerance, which is one of many problems in the filed of mathematical optimization and decision making theory has been considered. Also, the relevance of this work has been shown, as the conditions under which optimization under condition of tolerance can be applied. After that, the definitions of tolerant spaces and of conditions of tolerance has been given. Then, a scalar model of optimization under condition of tolerance has been considered. And, finally a two vector models of interest intersection - the one with artificial tolerance and the one with natural tolerance has been shown.

Key words: mathematical optimization, decision-making, vector optimization, decision-making under condition of tolerance .

1 Introduction

Recently, models of vector optimization received a lot of attention, as its formalism allows to substantially increase a level of adequacy in the real cases of decision-making. There are a lot of reasons for switching from scalar to vector criteria of optimization. It could be plurality of criteria (goals), also it could be plurality of subsystems of the integral system, or it might be dynamics of processes and different kinds of quality uncertainties [1, 8].

The investigator of the problem of vector optimization mostly agreed, that the number one natural step in this case must be selection of the Pareto set [4,5]:

$$X \backslash X^S = S^P, \qquad (1)$$

where X^S - is defined as a set of agreement, in which any solution can be improved without any losses, X^P - the Pareto set, and X - a set of alternatives.

Often, solution of the problem is limited by equation (1), and the Pareto set is presented as the optimal solution of the formal problem [5].

It should be noted, that in real cases, it’s frequently required either to find the only one optimal solution x^0, or to determine quite narrow optimal subset X^0. But till now, there is no a unified approach for solving this problem. We need to note, that it was proposed to use a zoning method for some cases [8].

This paper discusses another approach based on the ideas of a tolerant space. In addition, the introduction of conditions of tolerance puts forward a number of fundamentally new methods of choice.

2 Tolerant spaces and conditions of tolerance

”*Tolerance*” is a word, which can be understood widely. Literally it means ”*patience*”. In modern discourse, it can be interpreted as ”indifference, indistinguishability in different spheres of human activity.” In decision-making problems, and especially in vector optimization problems, tolerance plays a very important role. Here it comes to light in the process of formalizing specific tasks. Let’s introduce some definitions.

Definition 2.1. Let X be some random set, elements of which $x, x' \in X$ are so close to each other, so it is impossible to distinguish them. Hence, we will say, that x and x' are related with an relation of tolerance or are within a relationship of tolerance: $x \sim x'$.

Definition 2.2. Let X be some random set, elements of which $x,x' \in X$ are quite different, so they can be easily distinguished. Hence we will say, that x and x' are not tolerant and they are are outside the relation of tolerance, which means: $x \not\sim x'$.

Consequently, we can define the relation of tolerance as follows.

Definition 2.3. Tolerance τ is a binary relation of indistinguishability, which is given on a set of pairs. Then any set or space X, on which the relation of tolerance is determined τ, shall be called as tolerant space (X,τ).

Now, the intuitive idea behind the concept of tolerant spaces is clear enough. We can move in such a space within the limits of tolerance without noticing the difference.

Definition 2.4. The situation when the tolerance relation is defined in the formalized selection problem will be called the tolerance conditions, and, accordingly, the model is the optimization model under the tolerance conditions.

It should be noted, that the tolerance relation is not transitive. If $a \sim b$ and $b \sim c$, it does not necessarily result in $a \sim c$. The introduction of tolerance leads to a change in the structure of space, distorts the nature and properties of other relations, including transitivity.

Tolerance in vector optimization problems allows you to create a number of fundamentally new methods of solutions based on the ideas of nonindistinguishability, transform the known solution methods and remove the uncertainty of choosing the principle of optimality. In this case, both natural tolerance, determined from the conditions of the problem, and artificial, specially introduced for the implementation of specific principles of optimality in models, can be used.

To begin with, consider the simplest scalar model of choice in terms of tolerance.

3 Scalar optimization model under tolerance conditions

Let x be a solution, which is defined on set X. The quality of solution is rated by criteria $x \rightarrow y(x) \in R'$. There is a tolerance relation τ defined on R'. Then we have a model:

$$X^0 = X \cap \{x^0 | y(x^0) \geq \max_{x \in X} y(x) - \tau\}. \tag{2}$$

Thus, instead of a single optimal solution, we obtain an optimal subset of X^0 solutions that are equivalent under tolerance conditions τ:

$$x_1^0, x_2^0 \in X \Rightarrow x_1^0 \sim x_2^0 \Rightarrow |y(x_1^0) - y(x_2^0)| \leq \tau. \tag{3}$$

The composition of the order relation $\succ$ and the tolerance relation τ led to a new relation (the principle of optimality):

$$x^0 \succ x \Rightarrow y(x^0) \geq y(x) - \tau. \tag{4}$$

Often the tolerant approach is mixed with approximate methods for solving mathematical programming problems, when, in order to simplify the calculation procedure, a solution is searched for close to the optimal one. In this case, there is a ”*coarsening*” of the models and a departure from the exact optimum on the basis of the tolerance conditions corresponding to the statement of problems.

After we have considered scalar models in terms of tolerance, let’s move on to vector models.

4 Vector optimization models under conditions of tolerance

Let the quality of solutions $x \in X$ be rated by vector criteria:

$$x \rightarrow y(x) = \{y_j\}_{j \in J} \in R^m. \tag{5}$$

Also let the criteria space define the priority relation:

$$\lambda = \{\lambda j\} j \in J \in Rm, \tag{6}$$

and tolerance relation:

$$\tau = \{\tau j\} j \in J \in Rm, \tag{7}$$

where τ_j is the measure by j coordinate R. Thus, we get:

$$(X = \{x\}; y = \{x \rightarrow y_j(x)\}; \lambda = \{\lambda_j\}; \tau = \{\tau_j\}; j \in 1,...,m). \qquad (8)$$

The model (8) is a vector optimization model with priority under tolerance conditions and covers almost the entire class of vector optimization models.

Now let us consider the methods of searching for X^0, which are fundamentally possible only in conditions of tolerance.

4.1 Model of intersection of interests with artificial tolerance

For each j local criterion $y_j(x)$ and tolerance τ_j defined on it, local optimal subsets X_j^0 are determined and the interests of all criteria are matched by the intersection method:

$$X^0 = \cap X_j^0(y_j; \tau_j), \qquad (9)$$

where $X_j^0 = X \cap \{x^0 | y(x^0) \geq \max_{x \in X} y_j(x) - \tau_j\}$.

The idea behind this model is quite simple. Since each of the subsets X_j^0 consists of tolerant solutions that are optimal with respect to the j criterion, and we do not care which particular solution on X_j^0 will be chosen, then, naturally, the intersection of all X_j^0 gives the general consensus solution.

4.2 Model of intersection of interests with natural tolerance

Unfortunately, this method often does not work within the framework of the natural tolerance generated by the problem conditions, since the intersection $\cap X_j^0$ turns out to be empty. In this case, a model with artificial tolerance is applied, i.e. an increase in the level of tolerance is carried out, exceeding the natural level until the matching condition $\cap X_j^0 \neq \emptyset$ is satisfied, in this case we will have:

$$X \supset X_1^0 \supset X_2^0 \supset ... \supset X_m^0 \equiv X^0, X_1^0 = X \cap$$
$$\{x^0 | y_1(x) \geq \max y_1(x) - \tau_1\}, x \in X$$

$$X_j^0 = X_{j-1}^0 \cap \{x^0 | y_j(x^0) \geq \max y_1(x) - \tau_1\}, \quad (10)$$
$$X^0 \equiv X_m^0 = X_{m-1}^0 \cap \{x^0 | y_m(x^0) \geq \max y_m(x) - \tau_m\}.$$

Thus, at each j stage of the search due to tolerance, a local subset of tolerant solutions by the j criterion is determined - X_j^0, which is returned as an admissible set for the next most important criterion y_{j+1}. As a result, there is a sequential narrowing of the admissible set X to the optimal subset of X^0, taking into account the interests of all criteria, but subject to the principle of strict priority.

The model 10 can be used in the case of flexible priority of criteria, by transforming natural tolerance or by introducing artificial tolerance, taking into account the priority vector λ. However, in this situation, the difficult task of arguing the transformation arises $(\tau,\lambda) \rightarrow \tau^*$ according to the chosen compromise scheme.

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